Generic and object-oriented programming techniques for Monte Carlo simulation in C++

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Outline

1. Introduction
   - Background
   - Approach

2. Monte Carlo simulation
   - Generic algorithm implementation
   - Extensions for variance reduction
   - Early exercise

3. Varying the building blocks
   - Quasi-random Monte Carlo
   - Introducing stochastic interest rates
   - Simulating under the physical measure
Some observations

- Financial instruments and the quant models used to price them are increasingly complex.
- There is a multitude of alternative models, and being able to switch easily between models is a useful advantage.
- There is a great deal of commonality between initially disparate areas of application, e.g.
  - FX modelling analogy for inflation or commodities
  - Interest rates & zero coupon bond prices versus credit spreads & risk–neutral probabilities of default
  - The two “perspectives” of Black/Scholes/Merton
- Increasing need for models integrating several sources of risk.
As the complexity of software systems increased, “clean design” programming techniques were developed and refined, culminating in the now well–established discipline of software engineering.

- procedural programming
- generic and object–oriented programming
  - classes, objects and instances thereof
  - methods and message passing
  - abstraction: composition and inheritance
  - encapsulation
  - polymorphism
  - decoupling
  - templatisation
The observation that similar design problems recur in varying contexts led to the idea of common, reusable design patterns.

- Originally proposed as an architectural concept (Alexander et al. (1977) *A Pattern Language*)
- Established as a paradigm in software engineering (Gamma/Helm/Johnson/Vlissides (1995) *Design Patterns: Elements of Reusable Object–Oriented Software*)
- Playing an increasingly important role in quant finance applications (Joshi (2004) *C++ Design Patterns and Derivatives Pricing*)
Aim

Software that is
- reusable
- maintainable
- extensible
- satisfies the “implement once” principle

This is not always straightforward in quant finance applications, e.g.
- “model polymorphism” versus “payoff polymorphism”
- “generic programming” (templates) versus “object–oriented programming” (class hierarchies)
In this talk

- Pragmatic approach: use what works, rather than strictly adhering to one school or another
- Seek to create a collection of building blocks which can be varied independently
- Implement once (and in the right place) — doesn’t mean code once, and doesn’t preclude step-wise generalisation
- Important: figuring out the right way to decompose the problem
- Show that the design works when the building blocks are “mixed and matched”
Generic algorithm for Monte Carlo simulation

1. Initialise the variables
   RunningSum = 0
   RunningSumSquared = 0
   \( i = 0 \)

2. Generate the value of the stochastic process \( X \) at each date \( T_k \) relevant to evaluate the payoff.

3. Based on the values generated in step 2, calculate the payoff.

4. Add the payoff to RunningSum and the square of the payoff to RunningSumSquared.

5. Increment the counter \( i \).

6. If \( i \) is less than the maximum number of iterations, go to step 2.

7. Calculate the simulated mean by dividing RunningSum by the total number of iterations.

8. Calculate the variance of the simulations by dividing RunningSumSquared by the total number of iterations and subtracting the square of the mean.
Monte Carlo gatherer or accumulator — collects MC results for each draw and can be queried for the resulting MC estimate & relevant statistics.

```cpp
// instantiate object
MCGatherer my_gatherer;
...
// inside MC loop, say x contains the MC value for // that particular draw
my_gatherer += x;
...
// after MC loop, query mean and standard deviation
result = my_gatherer.mean();
stddev = my_gatherer.stddev();
```
Joshi (2004) presents an example of the MC gatherer in C++. A more sophisticated approach in C++ would be to use the `Boost.Accumulators template library` (see www.boost.org).

From the documentation:

*The library deals primarily with the concept of an accumulator, which is a primitive computational entity that accepts data one sample at a time and maintains some internal state.*
Template–based implementation of the generic MC algorithm

Template parameters:
- argument type
- return type
- random number generator type

Data members:
- random number generator instance
- functor (object representing a function) mapping the output of the random number generator to a realisation of the MC draw

RNG → Functor → MC draw result

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Monte Carlo simulation in C++
Role of the functor

Thus the functor encapsulates the mapping of the random numbers to the MC draw result.

In the typical quant finance context:

- Random numbers
- Model
- underlying values
- Payoff specification
- MC payoff realisation
Generic MC algorithm in C++ — class template

/** Template for generic Monte Carlo simulation.

   The template parameter class random_number_generator_type
   must implement a member function random(), which returns a
   realisation of the random variable of the type given by
   the template parameter class argtype.

   */

template <class argtype, class rettype,
            class random_number_generator_type>
class MCGeneric {
private:
   NormalDistribution N;
   // Functors mapping random variable to Monte Carlo payoff.
   boost::function<rettype (argtype)> f;
   random_number_generator_type random_number_generator;
public:
   inline MCGeneric(boost::function<rettype (argtype)> func,
                    random_number_generator_type& rng)
      : f(func), random_number_generator(rng) { }
   void simulate(MCGatherer<rettype>& mcgatherer,
                 unsigned long number_of_simulations);
   double simulate(MCGatherer<rettype>& mcgatherer,
                   unsigned long initial_number_of_simulations,
                   double required_accuracy,
                   double confidence_level = 0.95);
};
template <class argtype, class rettype,
         class random_number_generator_type>
void MCGeneric<argtype, rettype, random_number_generator_type>::
simulate(MCGatherer<rettype>& mcgatherer,
         unsigned long number_of_simulations)
{
    unsigned long i;
    for (i=0; i<number_of_simulations; i++)
        mcgatherer += f(random_number_generator.random());
}
Recap: implementation of the generic algorithm I

- The initialisations in Step 1 were done by the constructor of the `MCGatherer` object.

- The function call
  
  ```c++
  random_number_generator.random();
  ```

  generates the random variable in Step 2,

- to which the functor \( f \) is applied to calculate the payoff in Step 3. Note that \( f \) encapsulates the entire mapping from random variables to discounted payoff, thus the generic implementation of the Monte Carlo algorithm abstracts from the *stochastic model* (which maps basic random variables to the dynamics of the underlying assets) and any particular *financial instrument* (which maps the prices of underlying assets to a discounted payoff).
Recap: implementation of the generic algorithm II

- Step 4 is embodied in the overloaded `operator+=()` of the `MCGatherer` object.
- Steps 5 and 6 are managed by the `for` loop.
- Steps 7 and 8 are completed on demand by the member functions `mean()` and `stddev()` of `MCGatherer`.
The payoff specification mapping of a set of asset values at a set of time points to a **discounted payoff** can be encapsulated in a class.

This is advisable since the payoff mapping should not depend on how the asset values were generated.

Typically, asset values are generated by a class representing a stochastic process.

Note that except for the canonical case of simulating under the risk neutral measure under deterministic interest rates, the stochastic process must also supply the realisation of the numeraire.
Given a stochastic process and a payoff specification, it remains to link the two together.

This procedure is again independent of the specifics of the stochastic process and the payoff, thus can be accomplished generically by a template.
class MCMapping {
private:  // ...
public:
    /// Constructor.
    MCMapping(MCPayoff& xpayoff, price_process& xprocess, const TermStructure& xts, int xnumeraire_index = -1);
    /// Choose the numeraire asset for the equivalent martingale measure under which the simulation is carried out.
    bool set_numeraire_index(int xnumeraire_index);
    /// The function mapping a realisation of the (often multidimensional) random variable x to the discounted payoff.
    double mapping(random_variable x);
    /// The function mapping a realisation of the (often multidimensional) random variable x to multiple discounted payoffs.
    Array<double,1> mappingArray(random_variable x);
};
First extension: antithetic variates

Outer functor calls inner functor twice: once with the original variates, once with the antithetics, and averages the result.
Control variates can be implemented at different levels:

- Naive product control variates (e.g. geometric average option vs. arithmetic average option) can be encapsulated in a class derived from `MCPayoff`.

- Product control variates with (typically unknown) optimal weights require estimation from the simulation via (multivariate) regression.
  
  → implement at the `MCGatherer` level

- Control variates at the model level (i.e. using a simpler model permitting closed form solutions as a control variate) would involve two stochastic processes
  
  → implement at the `MCMapping` level (not yet implemented).
Single control variate

$f$ is the target functor, and $g$ is closely correlated, but with known expectation.

Monte Carlo estimate

\[ \hat{I}_N = \left( \frac{1}{N} \sum_{i=1}^{N} (f(u_i) - \beta g(u_i)) \right) + \beta E[g(U)] \]

An optimal choice of $\beta$ minimises $\text{Var}[f(U) - \beta g(U)]$, thus given by

\[ \beta = \frac{\text{Cov}[f(U), g(U)]}{\text{Var}[g(U)]} \]
Single control variate — C++

double MCGatherer<Array<double,1> >::CVestimate(int target, int CV, double CV_expectation) const
{
    double b = CVweight(target, CV);
    return mean(target) - b*(mean(CV) - CV_expectation);
}

double MCGatherer<Array<double,1> >::CVweight(int target, int CV) const
{
    double b = 0.0;
    if (CVon) {
        b = (covar(target, CV)/number_of_simulations() - mean(target)*mean(CV)) / (variance(CV)*number_of_simulations());
    }
    return b;
}
### Multiple control variates — power option

<table>
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<tr>
<th>Paths</th>
<th>MC value</th>
<th>CI width</th>
<th>CV MC value</th>
<th>CV CI width</th>
<th>Acc. gain</th>
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<td>0.101103</td>
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<td>23.01</td>
</tr>
</tbody>
</table>

Table 7.4 Pricing an option with payoff (7.66), strike 1.47876, power $\alpha = 0.1$ maturing in 1.5 years. Closed form value of the option is 0.101104. The model is Black/Scholes with initial stock price 100, 42.43% volatility and an interest rate of 5%. Pricing is done by “plain” Monte Carlo and with five standard call options with strikes 37.5, 50, 62.5, 75 and 125 as control variates. “CI width” refers to the width of a 95% confidence interval around the Monte Carlo estimate.
Multiple control variates — power option

Figure 7.3 plots the payoff of the power option and the payoff of a portfolio of standard options, where the portfolio weights are given by the (estimated) optimal control variate weights $\hat{\beta}_N$. Clearly, whenever a target payoff can be closely approximated by a portfolio of control variate payoffs, the portfolio and the target will be closely correlated, and the portfolio will be an effective control variate. Conversely, as Figure 7.3 illustrates, the regression (7.57) will converge for the portfolio payoff to optimally (in a probability–weighted least squares sense) approximate the target payoff.

7.7 Pricing instruments with early exercise features

At each point in time at which an option permits early exercise, the value of the option is given by the greater of the early exercise payoff and the continuation value (i.e., the value of not exercising early and instead holding on to the option). When using binomial lattice models (Chapter 3) or finite difference schemes (Chapter 5), one can price the option by backward induction, and thus the continuation value is known whenever the early decision needs to be evaluated. When simulating a Monte Carlo path, the continuation value is not known — herein lies the difficulty of pricing American and Bermudan options by Monte Carlo methods. However, in high–dimensional problems — be it due to a large number of underlying assets or due to the nature of the stochastic model — Monte Carlo simulation remains the primary numerical method.
Motivation

- Using binomial lattice models or finite difference schemes, one can price by backward induction, and thus the continuation value is known whenever the early decision needs to be evaluated.

- Simulating a Monte Carlo path, the continuation value is *not* known.

- The most common approach to approximate the continuation value is to estimate it by regression on a set of basis functions of state variables on a Monte Carlo path.

- This regression maps the values of the state variables at the time of potential early exercise to a (predicted) continuation value.
Lower bound on the true value

As the early exercise strategy only uses information that would be available to the option holder at the time of exercise, it can be no better than the best possible (i.e. optimal) strategy based on available information.

The quality of the lower bound depends on the choice of basis functions.
The lower bound algorithm: Notation

- Consider a set of financial variables $X_i, i = 1, \ldots, N$, observed at times $T_k, k = 0, \ldots, K$.
- Denote by $T_{\bar{k}(j)}, 0 \leq \bar{k}(j) \leq K$ the $j$-th time of potential exercise of an option.
- For each early exercise time $T_{\bar{k}(j)}, 0 \leq \bar{k}(j) < K$, approximate the continuation value by a linear combination of $n$ basis functions,

$$V(T_{\bar{k}(j)}) = \sum_{\ell=1}^{n} c_{\ell j} \psi_{\ell j}$$

(1)

$$\psi_{\ell j} := \psi_\ell \left(T_{\bar{k}(j)}, \{X_i(T_k)\}_{i=1}^{N}, k=0,\ldots,\bar{k}(j)\right), \quad \ell = 1, \ldots, n$$
The lower bound algorithm I

Following Longstaff and Schwartz (2001), estimate \( c_{\ell j} \) for each \( T_{\overline{k}(j)} \) by

**Step 1:** Generate \( n_{\text{paths}} \) simulated Monte Carlo paths of the financial variables \( X_i \) observed at times \( T_k \). (These paths will be used to estimate the \( c_{\ell j} \) and then discarded.)

**Step 2:** Initialise a vector \( y \) of length \( n_{\text{paths}} \) with the option payoff at time \( T_K \) on each path, discounted to the last time point \( T_{\overline{k}(J)} < T_K \) of potential early exercise. Set the “current” time \( t = T_{\overline{k}(j)} \). The initial value of the index \( j \) is \( J \) (i.e. we will be proceeding by backward induction).

**Step 3:** Calculate a vector \( e \) of length \( n_{\text{paths}} \), containing the payoff on each path if the option is exercised at time \( t \).
Step 4: Determine the $c_{\ell j}$ for all $\ell = 1, \ldots, n$ and the current $j$ by least squares fit, i.e. choose the $c_{\ell j}$ to minimise

$$\sum_{h=1}^{n_{\text{paths}}} (y_h - V^{(h)}(T_{\bar{k}(j)}))^2$$

where $V^{(h)}(T_{\bar{k}(j)})$ is given by (1) calculated using the realisations of $X_i(T_k)$ on the $h$-th path.

Step 5: Using the thus determined $c_{\ell j}$, set each $y_h$ to $\max(e_h, V^{(h)}(T_{\bar{k}(j)}))$ discounted to time $T_{\bar{k}(j-1)}$.

Step 6: If $T_{\bar{k}(j)}$ is the earliest time of potential early exercise, stop. Otherwise, set $j = j - 1$ and $t = T_{\bar{k}(j)}$, and go to Step 3.
Longstaff & Schwartz’s example

```c++
Array<double,2> paths(8,4);
paths = 1.00, 1.09, 1.08, 1.34,
       1.00, 1.16, 1.26, 1.54,
       1.00, 1.22, 1.07, 1.03,
       1.00, 0.93, 0.97, 0.92,
       1.00, 1.11, 1.56, 1.52,
       1.00, 0.76, 0.77, 0.90,
       1.00, 0.92, 0.84, 1.01,
       1.00, 0.88, 1.22, 1.34;

Array<double,1> T(4);
T = 0.0, 1.0, 2.0, 3.0;
double K = 1.10;
Payoff put(K,-1);
boost::function<double (double)> f;
f = boost::bind(std::mem_fun(&Payoff::operator()),&put, _1);
FlatTermStructure ts(0.06,0.0,10.0);
Array<double,3> genpaths(8,4,1);
genpaths(Range::all(),Range::all(),0) = paths;
Array<double,2> numeraire_values(8,4);
for (i=0;i<4;i++) numeraire_values(i,Range::all()) = 1.0/ts(T(i));
boost::function<double (double,const Array<double,1>&)> payoff = boost::bind(LSArrayAdapter, _1,_2,f,0);
```

• a `std::vector` of `boost::function<double(double t, const Array<double,1>& finvars)>`
vector<boost::function<double (double, const Array<double, 1>&)>> basisfunctions;
int degree = 2;
Array<int, 1> p(1);
for (i=0; i<=degree; i++) {
    p(0) = i;
    add_polynomial_basis_function(basisfunctions, p);
}
LongstaffSchwartzExerciseBoundary
genls(T, genpaths, numeraire_values, payoff, basisfunctions);
Array<double, 2> genpath(4, 1);
Array<double, 1> num(4);
for (i=0; i<8; i++) {
    genpath = genpaths(i, Range::all(), Range::all());
    num = numeraire_values(i, Range::all());
    MCestimate += genls.apply(genpath, num);
}
std::cout << MCestimate.mean() << std::endl;
Early exercise and the generic MC templates

```cpp
LSExerciseStrategy<LongstaffSchwartzExerciseBoundary>
exercise_strategy(boundary);
MCMapping<GeometricBrownianMotion,Array<double,2> >
mc_mapping(exercise_strategy,gbm,ts,numeraire_index);
boost::function<double (Array<double,2>)> func =
boost::bind(&MCMapping<GeometricBrownianMotion,
               Array<double,2> >::mapping,
               &mc_mapping,_1);
MCGeneric<Array<double,2>,double,
          RandomArray<ranlib::NormalUnit<double>,double> >
mc(func,random_container);
MCGatherer<double> mcgatherer;
size_t n = minpaths;
boost::math::normal normal;
double d = boost::math::quantile(normal,0.95);
cout << "Paths,MC value,95% CI lower bound,95% CI upper bound"
    << endl;
while (mcgatherer.number_of_simulations()<maxpaths) {
    mc.simulate(mcgatherer,n);
    cout << mcgatherer.number_of_simulations() << ','
         << mcgatherer.mean() << ','
         << mcgatherer.mean()-d*mcgatherer.stddev() << ','
         << mcgatherer.mean()+d*mcgatherer.stddev() << endl;
    n = mcgatherer.number_of_simulations(); }
```
### Lower bounds with different basis functions

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<tr>
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<th>MC price</th>
<th>95% conf. bounds</th>
<th>MC price</th>
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**Incl. European put**

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The pseudo–random Monte Carlo estimate converges with $O(N^{-\frac{1}{2}})$.

Quasi–random Monte Carlo seeks to improve on this convergence by deterministically filling the unit hypercube “more evenly.”

Acceleration of convergence to nearly $O(N^{-1})$ — herein lies the difference to the variance reduction techniques, which only affected the implicit constant in $O(N^{-\frac{1}{2}})$ convergence.
Filling “more evenly”

Figure 7.4 A grid of points in the unit square: the dashed box is completely empty of points

Figure 7.5 Better coverage of the unit square

For example, suppose one needs to generate 1,000 paths of geometric Brownian motion observed at 100 time steps, so $N$ is 1,000 and $d$ is 100. In pseudo-random Monte Carlo, one can generate 100,000 (i.e. $N \times d$) independent univariate uniform random variates, which are transformed into independent, normally distributed Brownian motion increments and then assembled (in batches of 100) to produce 1,000 paths. In quasi-random Monte Carlo, one has to generate a sequence of 1,000 quasi-random variates of dimension 100, i.e. one for each path — using 100,000 quasi-random variates of dimension 1 would lead to an incorrect result. As the rate of convergence of quasi-random methods tends to deteriorate in higher dimensions, it is recommended to keep the dimension $d$ as low as possible, for example by using the predictor/corrector method whenever necessary and possible.
Pseudo-random vs. quasi-random II

- Quasi–random Monte Carlo explicitly depends on the problem dimension.
- In pseudo–random Monte Carlo a single univariate random number generator was sufficient to generate all the required independent random variates for a sample path.
- In quasi–random Monte Carlo, two consecutive elements of the quasi–random sequence are not independent.
One must distinguish between the set of random variates making up a single random vector \( u_i \) required to produce a realisation of the simulation \( f(u_i) \), and consecutive realisations of the random vector \( u_i \) drawn \( N \) times to calculate the Monte Carlo estimate.

The dimension \( d \) of the \( u_i \) is the required dimension of the quasi–random sequence, while \( N \) is the length of the sequence.
It is difficult to obtain a meaningful error estimate through quasi–random methods alone, but a particularly simple way of getting around this problem is to randomise (in a pseudo–random sense) over a set of quasi–random simulation runs.

A simple way a quasi–random sequence can be randomised is by shifting each point in the sequence by the same random amount $v$, uniform on $[0, 1]^d$. 
Coordinate values greater that one are “wrapped” using modulo–1 division, i.e. if $u_{i}^{(j)}$ is the $j$–th coordinate of the $i$–th element in the original sequence, its randomised counterpart $\hat{u}_{i}^{(j)}$ is given by

$$\hat{u}_{i}^{(j)} = (u_{i}^{(j)} + v^{(j)}) \mod 1$$

where $v^{(j)}$ is the $j$–th coordinate of $v$. 
We now have

- quasi–random Monte Carlo (simulation using the sequence $\hat{u}$) nested inside pseudo–random Monte Carlo (repeated draws of $v$)

- a meaningful standard deviation of the Monte Carlo estimate (which is now an average over the different quasi–random Monte Carlo estimates for each $\hat{u}$) can be obtained at the level of the outer loop.
**Convergence example**

Figure 7.6  *Convergence of confidence intervals for Monte Carlo estimate of Margrabe option price, log/log scale, trendline added for quasi-random case.*
Example with control variates

Figure 7.7 Convergence of confidence intervals for Monte Carlo estimate of Margrabe option price, with and without control variates (CV), log/log scale, trendline added for quasi-random (QR) cases.
Up to now, the simulations were implemented using the stochastic process class `GeometricBrownianMotion` (representing a multi–asset Black/Scholes model under deterministic interest rates).

This can be replaced by a class `GaussMarkovWorld` (representing a multi–asset, multi–currency Gauss/Markov Heath/Jarrow/Morton model).

It is convenient to simulate zero coupon bond prices and forward asset prices and exchange rates, meaning that payoffs have to be appropriately defined.
All other building blocks remain unchanged (e.g. MCGatherer, MCGeneric, MCMapping, the Longstaff/Schwartz implementation, etc.).

Extension to the LIBOR Market Model (not yet implemented) would only require the implementation of the appropriate class to replace GaussMarkovWorld.
### Table 8.3 Suboptimal values for an American put option based on Longstaff/Schwartz exercise strategies with different polynomial basis functions and different numbers of training paths.

The MC values are based on 3276800 paths with 60 time steps on each path; the 95% confidence intervals are \( \pm 0.07 \) around the given value. The price of the corresponding European option is 46.66.

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<th>MC value</th>
<th>Max. polynomial deg.</th>
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Martingale measures and real–world probabilities

Well–known results of Financial Economics:

- Any asset with a strictly positive price process is a valid numeraire.
- For every numeraire $S$, in an arbitrage–free market there exists at least one probability measure $Q_S$ equivalent to the real–world measure $P$, such that all (traded) asset prices discounted by $S$ are $Q_S$–martingales.
The Numeraire Portfolio

Perhaps less well–known, the converse is also true:

For every probability measure $Q_H$ equivalent to the real–world measure $P$, there exists a self–financing portfolio strategy with a value process $H$ that is strictly positive almost surely, such that all (traded) asset prices discounted by $H$ are $Q_H$–martingales.

Note that

Since $P$ is trivially equivalent to $P$, there exists a portfolio strategy $\hat{H}$ such that all (traded) asset prices discounted by $\hat{H}$ are martingales under the real–world measure. $\hat{H}$ is commonly called the numeraire portfolio or deflator.
Properties of the Numeraire Portfolio

- If there are no redundant assets, the numeraire portfolio is unique.
- The numeraire portfolio exists even in an incomplete market — however, only *traded* asset prices discounted by $\hat{H}$ are necessarily martingales under $P$.
- Adding further, non-redundant assets typically changes the composition of the numeraire portfolio.
- When the numeraire portfolio is defined as (its density of) weights in the Arrow/Debreu securities, these weights can be identified with the real–world probability measure.
- $\hat{H}$ maximises the expected growth rate of the portfolio value and the expected logarithm of terminal wealth. It minimises the expected time needed to reach a given wealth level for the first time.
Assume that asset dynamics are given by

\[ dS_i(t) = S_i(t)(\mu_i dt + \sigma^{(i)} dW(t)) \]

where \( W(t) \) is a \( d \)-dimensional Brownian motion under the real-world measure.

Switching numeraires from the numeraire portfolio \( \hat{H}(T) \) to the continuously compounded savings account \( \beta(T) \), we have

\[ \frac{dP}{dQ_\beta}_{\mathcal{F}_T} = \frac{\hat{H}(T)\beta(0)}{\hat{H}(0)\beta(T)} \]
This is the $Q_\beta$–martingale

$$\exp \left\{ \int_0^t \sigma_H dW_\beta(t) - \frac{1}{2} \int_0^t \sigma_H^2 dt \right\}$$

Thus by Girsanov’s theorem $P$ and $Q_\beta$ are related via

$$dW(t) = dW_\beta(t) - \sigma_H dt$$

and we can write

$$dS_i(t) = S_i(t)(\mu_i dt + \sigma^{(i)}(dW_\beta(t) - \sigma_H dt))$$

We must therefore have

$$\mu_i - \sigma_H \sigma^{(i)} = r$$

and we may interpret $\sigma_H$ as the market price of risk.
Implementation

This is implemented in the C++ code, though not explicitly discussed in the book:

GaussMarkovWorld implements a member function

    set_numeraire(int num);

which allows the selection of a simulated asset as the numeraire (and thus the stochastic dynamics are generated under the appropriate measure, with the appropriate drifts).

Selecting as the numeraire an asset with a volatility (vector) equal to the market price of risk results in simulation under the physical measure.
This talk was mainly based on this book:

For more information, and to download the C++ code (released under a Modified BSD License), visit:

www.schlogl.com/QF